General Relativity $\boxed{\text{Week 2}}$

Last time:

Lorentzian manifold (M, g): Smooth manifold with an assignment $p \longrightarrow g_p$ of a Lorentzian inner product on T_pM $\forall p \in M$ such that $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ is smooth $\forall$ local coordinate chart

- A vector field $X \in \Gamma(M)$ is timelike / null / spacelike
  if $X_p \in T_pM$ is " " " $\forall p \in M$
- A submanifold $S \subseteq M$ is timelike / null / spacelike
  if $T_pS \subseteq T_pM$ is " " " $\forall p \in S$
- A C^1 curve γ is timelike / null / spacelike
  if $\dot{\gamma}$ is . . . .

• In local coordinates: $g = g_{ij}\, dx^i dx^j$

Examples:

- Minkowski spacetime: $(\mathbb{R}^{n+1}, \eta)$

  In Cartesian coordinates: $\eta = -(dx^0)^2 + \dots + (dx^n)^2$
  (constant components)

  So $\left\{\frac{\partial}{\partial x^a}\right\}_{a=0}^{n}$: Orthonormal basis at every $p \in \mathbb{R}^{n+1}$

timelike curve stays inside the timecone based at every point. ①

The straight line $\gamma: t \longrightarrow (t, vt)$, $v \in \mathbb{R}^n$ → timelike, if $|v| < 1$; null, $|v| = 1$; spacelike, $|v| > 1$.

Using ①: I can extend the definition of a timelike curve in $(\mathbb{R}^{n+1}, \eta)$ to C^0 curves. cone condition $\Rightarrow$ Lipschitz condition
$\Rightarrow$ almost everywhere differentiable

- If $(\bar{M}, \bar{g})$ is a Riemannian manifold,

  $(M = \mathbb{R} \times \bar{M},\ g = -dt^2 + \bar{g})$ is a Lorentzian manifold

  $t: M \to \mathbb{R}$: Projection on the 1st factor

  - Translation in t: Isometries

- ~~Conformal~~ If (M, g) is a Lor. manifold, and $f > 0$: $(M, f \cdot g)$ is also one, $\tilde{g} = f \cdot g$: conformal to g (same null cone at every point)

Remarks

- Change of coordinates:

  If $g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta = g_{\alpha\beta}\, dx^\alpha dx^\beta$ in $(x^0, \dots, x^n)$

  And $(x^0, \dots, x^n) \longrightarrow (y^0, \dots, y^n)$

  Then $g = \tilde{g}_{\alpha\beta}\, dy^\alpha dy^\beta$

  Since $dy^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} dx^\mu$

  $$\Rightarrow \quad g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \tilde{g}_{\alpha\beta} \frac{\partial y^\beta}{\partial x^\nu}$$

  $$\Leftrightarrow \quad \tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha} g_{\mu\nu} \frac{\partial x^\nu}{\partial y^\beta}$$

  (or in matrix form:

  $$[\tilde{g}] = \left[\frac{\partial x}{\partial y}\right]^T [g] \left[\frac{\partial x}{\partial y}\right]$$

  Recall $\left[\frac{\partial y}{\partial x}\right]\left[\frac{\partial x}{\partial y}\right] = I$)

- Not every manifold admits a Lorentzian metric (unlike the Riemannian case)

  Topological restrictions: M must admit a line subbundle

  (If M compact: $\chi(M) = 0$)

  (We will see it in the exercises)

Length of a curve:

- If γ is causal: $\ell(\gamma) := \int_a^b \sqrt{-g(\dot\gamma,\dot\gamma)}\,dt$
- If γ is spacelike: $\ell(\gamma) := \int_a^b \sqrt{+g(\dot\gamma,\dot\gamma)}\,dt$

Make sense for piecewise C^1 curves

We will not talk about the length of mixed curves.

<u>Note</u>: Length is independent of reparametrization

If $h:(c,d)\to(a,b)$ is piecewise C^1 and increasing:

$$\ell(\gamma\circ h) = \int_c^d \sqrt{-g(\dot\gamma(h(s))\,h'(s),\dot\gamma(h(s))\cdot h'(s))}\,ds \overset{t=h(s)}{=} \int_a^b \sqrt{-g(\dot\gamma(t),\dot\gamma(t))}\,dt$$

$$= \ell(\gamma)$$

$$\left[(\gamma\dot{\circ} h)(s) = \dot\gamma(h(s))\cdot h'(s)\right]$$

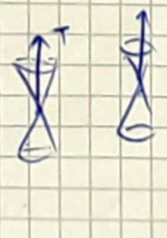
Time orientation:

Let (M,g) be a Lorentzian manifold

$\forall p\in M$: $I_p = \{v\in T_pM : g(v,v)<0\}$ has two components, $I_p^\pm$. Fixing a $T\in I_p$:

We can define $I_p^+ = \{v\in I_p : g(T,v)<0\}$.

<u>Question</u>: Can we consistently choose I_p^+ $\forall p\in M$ in a continuous way?



On Minkowski: Just choose $T = \frac{\partial}{\partial x^0}$

Def: A <u>time orientation</u> of (M,g) is an assignment $\forall p\in M$ of a future timecone I_p^+ which is smooth in the following sense: $\forall p\in M$, $\exists$ smooth vector field V in a neighborhood of p such that $V|_q \in I_q^+$ for q in that neighborhood.

If (M,g) admits a time orientation $\Rightarrow$ time orientable.

A time oriented (M,g): <u>spacetime</u>.

Lemma: (M,g) is time orientable if <u>and only</u> if a globally timelike vector field T exists on M.

Proof:

"$\Leftarrow$": Let $T\in \Gamma(M)$ be timelike everywhere.

Define $\forall p\in M$: $I_p^+ = \{v\in I_p : g(T,v)<0\}$

Then T satisfies the definition.

"$\Rightarrow$": Let I_p^+ be the chosen assignment.

Let $\{U_a\}_{a\in A}$ be a collection of open sets on M s.t $\bigcup_{a\in A} U_a = M$ and $\forall a\in A$:

$\exists$ vector field V_a on U_a s.t $V_a|_q \in I_q^+$ $\forall q\in U_a$.

Let $\{\chi_\beta\}_{\beta\in B}$ be a <u>partition of unity</u> subordinate to $\{U_a\}_{a\in A}$, i.e. $\forall \beta\in B$, $\exists a=a(\beta)$ s.t

$\operatorname{supp}\chi_\beta \subseteq U_{a(\beta)}$

Set $V := \sum_{\beta} \chi_\beta \cdot V_{a(\beta)}$

Sum is well-defined: $\chi_\beta \cdot V_{a(\beta)}$ is well-defined on all of M, and the sum is finite around every point.

(so V: smooth vector field)

We will show that V is everywhere timelike:

$$\forall p \in M: \quad V_{a(\beta)}\big|_p \in \underbrace{I_p^+ \cup \{0\}}_{\substack{\text{convex cone,} \\ \text{at least one non-zero}}} \quad \Rightarrow \quad \sum_{\beta} \chi_\beta \cdot V_{a(\beta)} \in I_p^+ .$$

$\square$

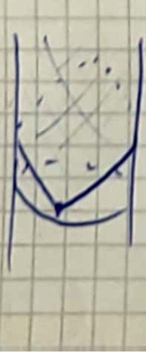
Def: Let (M, g) be time oriented.

- Timelike future $\underset{\subseteq M}{I^+(p)} = \{ q \in M : \exists \gamma : [0,1] \to M,\ \gamma(0) = p,\ \gamma(1) = q,\ \dot{\gamma}(t) \in I^+_{\gamma(t)}\ \forall t \in [0,1] \}$

- Causal future: $J^+(p) = \dots$

Ex.

- On Minkowski: $I^+(0) = I_0^+$ if we identify $\mathbb{R}^{n+1}$ with $T_0 \mathbb{R}^{n+1}$
- On $\mathbb{R} \times S^1$, $g = -dt^2 + dx^2$



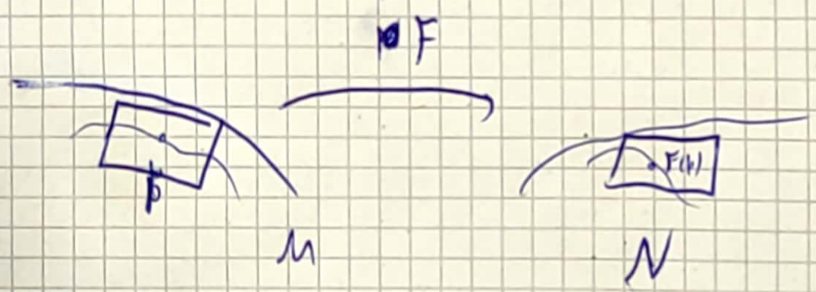
Note: • Time orientability $\neq$ topological orientability

• In general: $I^+(p)$ can contain a whole neighborhood of p

(check $S^1 \times S^1$, $g = -dt^2 + dx^2$)

But restricting g to a small enough neighborhood U of p: Similar picture as in Minkowski.

A few more stuff from the theory of manifolds:



Let $F: M \to N$ be smooth.

$dF: T_pM \longrightarrow T_{F(p)}N$: The differential of F.

If γ is a curve through p in M:

$$dF(\dot{\gamma}) = (\dot{F \circ \gamma})$$

In local coordinates: $\left(dF\left(\frac{\partial}{\partial x^i}\right)\right)^a = \frac{\partial F^a}{\partial x^i}$

So $(dF(X))^a = \frac{\partial F^a}{\partial x^i} X^i$

Def: 1) F is an immersion if $dF|_p : T_pM \to T_pN$ is injective $\forall p \in M$

2) embedding if immersion & 1-1 & homeomorphism on its image

3) Diffeomorphism if bijective & F, F^{-1} embeddings

Pull-back of a metric

Def: Let $F: M \longrightarrow (N,h)$ be an immersion

Pull-back of h via F:

$$g = F^* h, \quad g(X,Y) = h(dF(X), dF(Y))$$

In local coordinates: $g_{\alpha\beta} = \frac{\partial F^\mu}{\partial x^\alpha} \frac{\partial F^\nu}{\partial x^\beta} h_{\mu\nu}$

- If h Riemannian $\longrightarrow$ F^*h Riemannian
- If h Lorentzian, F^*h is not necessarily even non-degenerate.

Def: $F: (M,g) \to (N,h)$ is

- A local isometry if $dF_p : (T_pM, g_p) \to (T_{F(p)}N, h_{F(p)})$ is a linear isometry
- A global isometry if it is a local isometry and bijective.

- All global isometries of $(\mathbb{R}^{n+1}, \eta)$ to itself are linear maps, belonging to $O(1,n) \ltimes \mathbb{R}^{n+1}$

Tensors and tensor products

Let M be a smooth manifold and $p \in M$.

Def: The space of (k,l) tensors at p:

Set of multilinear maps

$$T: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{l} \longrightarrow \mathbb{R}$$

Tangent vector: Type (1,0) $\quad v: T_p^* M \to \mathbb{R}$

Co-tangent vector: (0,1)

Lorentzian metric: (0,2)

<u>Tensor product</u>: $\left.\begin{array}{l} \omega_1 : (k_1, l_1) \\ \omega_2 : (k_2, l_2) \end{array}\right\} \quad \omega_1 \otimes \omega_2 : \ (k_1+k_2,\ l_1+l_2)$

$$\omega_1 \otimes \omega_2 (y_1, \dots, y_{k_1+k_2}, v_1, \dots, v_{l_1+l_2}) =$$

$$= \omega_1 (y_1, \dots, y_{k_1}, v_1, \dots, v_{l_1}) \cdot \omega_2 (y_{k_1+1}, \dots y_{k_1+k_2}, v_{l_1+1}, \dots$$

Not commutative!

In local coordinates $(x^0, \dots, x^n)$:

Basis for tensors of type (k, l):

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l} \right\}_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_l}=0}^{n}.$$

Components of tensor:

$$T = \underbrace{T^{i_1 \dots i_k}_{j_1 \dots j_l}}_{} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$$

$\to$ contravariant indices

$\to$ covariant indices

Change of coordinates: $(x^0, \dots, x^n) \longrightarrow (y^0, \dots, y^n)$

$$\tilde{T}^{i_1 \dots i_k}_{j_1 \dots j_l} = T^{a_1 \dots a_k}_{\beta_1 \dots \beta_l} \frac{\partial y^{i_1}}{\partial x^{a_1}} \dots \frac{\partial y^{i_k}}{\partial x^{a_k}} \cdot \frac{\partial x^{\beta_1}}{\partial y^{j_1}} \dots \frac{\partial x^{\beta_l}}{\partial y^{j_l}}$$